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Overstability of a viscoelastic fluid layer heated from below

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The stability of a horizontal layer of Maxwellian fluid heated from below is considered. Critical Rayleigh numbers, wave-numbers, and frequencies for overstability are determined for both free and rigid boundaries. Elasticity is found to destabilize the fluid, and the presence of rigid boundaries is found to be slightly stabilizing.

1. Introduction

The problem of the onset of thermal instability in a horizontal layer of viscous fluid heated from below has its origin in the experimental observations of Bénard (1900). The contributions of many who have subsequently studied this phenomenon are discussed in detail in the monograph by Chandrasekhar (1961). For this problem the 'principle of exchange of stabilities' is valid, so the instability is manifested as a steady, cellular, convective motion.

More recently, the effects of rotation of the fluid layer (Chandrasekhar 1953; Chandrasekhar & Elbert 1955), imposition of a magnetic field (Chandrasekhar 1952, 1954) and mass diffusion (Stern 1960; Sani 1965) on the stability of thermally stratified fluids have been studied. In each of these cases it is found that in certain ranges of the governing parameters the fluid layer becomes overstable, i.e. the thermal instability gives rise to an oscillatory convective motion. Overstability is possible in the presence of rotation or a magnetic field because they lend an elastic-like behaviour to the fluid thereby enabling it to sustain appropriate modes of wave propagation. It is therefore expected that a layer of viscoelastic fluid can become overstable due solely to heating from below.

The purpose of the research reported here is to evaluate the conditions under which thermally induced overstability occurs in a viscoelastic fluid. The only previous work which deals directly with thermal instability of a viscoelastic fluid appears to be that of Herbert (1963) who studied plane Couette flow heated from below. He found that a finite elastic stress in the undisturbed state is necessary for elasticity to affect stability. This is true, however, only because a stationary mode of disturbance was assumed—the argument is not valid in the case of overstable modes. The closely related problem of overstability in isothermal cylindrical Couette flow has been considered by Beard, Davies & Walters (1966). Their investigation indicated that overstability is to be expected in moderately elastic

Maxwell fluids, with the critical Taylor number decreasing with increasing elasticity of the fluid. Several other papers which deal only with the stationary stability of viscoelastic cylindrical Couette flow are omitted from reference here; however, a comparative review of the literature on this problem may be found in the study by Miller (1967).

2. Formulation

Consider a layer of fluid confined between two horizontal planes separated by a distance d. The upper plane is maintained at a temperature T_1 which is lower than T_0, the temperature of the bottom plane. The fluid is assumed to have a viscoelastic nature described by the Maxwell constitutive relation. This rather idealized constitutive relation is deemed sufficient to reveal the basic effects of viscoelasticity on thermal instability, particularly in view of the extremely low shear rates involved and the linearization process utilized in the analysis. The equations which govern the behaviour of a Maxwell fluid are

$$\rho(\partial_t v_k + v_j \partial_j v_k) = \rho F_k + \partial_j T_{jk}, \tag{1}$$

$$\rho(\partial_t e + v_j \partial_j e) = T_{jk} V_{jk} - \partial_j q_j, \tag{2}$$

$$\partial_t \rho + \partial_j(\rho v_j) = 0, \tag{3}$$

$$q_j = -k\partial_j T, \tag{4}$$

$$\tau_{ij} + t_0(\partial_t + v_k \partial_k)\tau_{ij} = \mu(\partial_j v_i + \partial_i v_j), \tag{5}$$

where v_i is the velocity vector, F_i the body force per unit mass, T_{ij} the stress tensor, V_{ij} the rate of strain tensor, e the total energy per unit mass, q_i the heat flux vector, T the temperature and τ_{ij} the stress deviator $T_{ij} - P\delta_{ij}$. The fluid density and viscosity are denoted by ρ and μ respectively, and t_0 is the Maxwell relaxation time.

The initial quiescent state of the fluid is described by

$$\bar{v}_i = 0, \quad \bar{\tau}_{ij} = 0, \quad \bar{T} = T_0 + \beta x_3, \tag{6}$$

where $\beta = (T_1 - T_0)/d$. Since the body force and density field in the fluid are $F_i = (0, 0, -g)$ and $\rho = \rho_0[1 - \gamma(T - T_0)]$ respectively, thermally induced motion of the fluid is governed by the equations

$$(\partial_t + v_j \partial_j) v_k = -\rho_0^{-1}\partial_k p' + \lambda_k g\gamma\theta + \rho_0^{-1}\partial_j \tau_{jk}, \tag{7}$$

$$(\partial_t + v_j \partial_j)\theta = \kappa \partial_j \partial_j \theta, \tag{8}$$

$$\partial_j v_j = 0, \tag{9}$$

which are derivable from the preceding equations with the Boussinesq approximation and the assumption that viscous dissipation can be neglected. Here p' is the pressure relative to its hydrostatic value, $\theta = T - T_0$, γ is the thermal expansivity of the fluid, κ is its thermal diffusivity, and $\lambda_k = (0, 0, 1)$.

Considering, in the usual manner, a small perturbation about the initial configuration of the fluid,

$$v_k = v_k, \quad p = p', \quad \theta = \beta x_3 + \theta', \quad \tau_{jk} = \tau_{jk}, \tag{10}$$

the linearized equations governing the perturbation quantities are readily found to be

$$\partial_t v_k = -\rho_0^{-1}\partial_k p' + \lambda_k g\gamma\theta' + \rho_0^{-1}\partial_j \tau_{jk}, \tag{11}$$

$$\partial_t \theta' + \lambda_j v_j \beta = \kappa \partial_j \partial_j \theta' \tag{12}$$

and

$$\partial_j v_j = 0. \tag{13}$$

Now the linearized form of the constitutive relation (5) can be substituted into (11) after having operated on it by $(1 + t_0\partial_t)$. If $\partial_l \partial_l$ of the third component of the resulting equation is combined with ∂_3 of the divergence of the same equation, the following relation is obtained:

$$(1 + t_0\partial_t)[\partial_t \nabla^2 v_3 - g\gamma\nabla_1^2\theta'] = \nu\nabla^4 v_3, \tag{14}$$

where $\nabla_1^2 = \partial_1\partial_1 + \partial_2\partial_2$ and ν is the kinematic viscosity. Introducing the following dimensionless variables:

$$\left.\begin{aligned} (x,y,z) &= (x_1/d, x_2/d, x_3/d), \quad \tau = \kappa t/d^2, \\ (u,v,w) &= (v_1 d/\kappa, v_2 d/\kappa, v_3 d/\kappa), \quad T' = \theta'/\Delta T, \end{aligned}\right\} \tag{15}$$

the governing equations (14) and (8) can be written as

$$(1 + \Gamma\partial_\tau)(P^{-1}\partial_\tau\nabla^2 w - R\nabla_1^2 T') = \nabla^4 w \tag{16}$$

and

$$(\partial_\tau - \nabla^2) T' = -w, \tag{17}$$

subject to

$$w = \partial w/\partial z = \theta' = 0 \quad \text{at a rigid boundary,} \tag{18}$$

or

$$w = \partial^2 w/\partial z^2 = \theta' = 0 \quad \text{at a free boundary.} \tag{19}$$

Here P is the Prandtl number, ν/κ, R the Rayleigh number, $g\gamma\beta d^4/\nu\kappa$ and $\Gamma = t_0\kappa/d^2$ is an elastic parameter which may be interpreted as a Fourier number in terms of t_0.

Following the usual approach, perturbations of the form

$$w = W(z)\exp[i(ax+by)+\sigma\tau]$$

and

$$T' = T(z)\exp[i(ax+by)+\sigma\tau]$$

are considered. Equations (16) and (17) then become

$$(1+\Gamma\sigma)[\sigma P^{-1}(D^2-\alpha^2)W + R\alpha^2 T] = (D^2-\alpha^2)^2 W, \tag{20}$$

$$[\sigma - (D^2-\alpha^2)]T = -W, \tag{21}$$

where $\alpha^2 = a^2 + b^2$. Clearly, as $\Gamma \to 0$ the foregoing equations approach those governing the classical Bénard problem.

3. Solution

In this section solutions are presented for the cases of free and rigid boundaries. The former, although difficult to realize experimentally, is of importance since its exact solution is readily obtained. Furthermore, it is shown *a posteriori* that the critical Rayleigh numbers for the two cases differ by only a small amount.

3.1. *Free boundaries*

If both boundaries are plane and free of viscous stresses, the problem is governed by (20) and (21) subject to the boundary conditions

$$W = D^2W = T = 0 \quad \text{at} \quad z = 0, 1. \tag{22}$$

Equations (20) and (21) can readily be combined to yield

$$(D^2-\alpha^2)(D^2-\alpha^2-\sigma)[D^2-\alpha^2-P^{-1}(1+\Gamma\sigma)\sigma]\,W = (1+\Gamma\sigma)R\alpha^2 W. \tag{23}$$

Examination of boundary conditions (22) and equation (23) indicates that the required solution is

$$W = W_0 \sin n\pi z \quad (n = 1, 2, 3, \ldots). \tag{24}$$

The characteristic equation is therefore

$$(n^2\pi^2+\alpha^2)(n^2\pi^2+\alpha^2+\sigma)[n^2\pi^2+\alpha^2+P^{-1}(1+\Gamma\sigma)\sigma] = -(1+\Gamma\sigma)R\alpha^2. \tag{25}$$

This equation can be rearranged as

$$\sigma^3+\frac{1}{\Gamma}[\Gamma(n^2\pi^2+\alpha^2)+1]\,\sigma^2+\left[\frac{1+P}{\Gamma}(n^2\pi^2+\alpha^2)+\frac{PR\alpha^2}{n^2\pi^2+\alpha^2}\right]\sigma + \frac{P}{\Gamma}\left[(n^2\pi^2+\alpha^2)^2+\frac{R\alpha^2}{n^2\pi^2+\alpha^2}\right] = 0, \tag{26}$$

which may be denoted symbolically as

$$\sigma^3+A_1\sigma^2+A_2\sigma+A_3 = 0.$$

From the elementary theory of algebraic equations it is clear that a neutral oscillatory mode (i.e. $\sigma = i\sigma_i$) occurs if

$$A_3 > 0, \quad A_1A_2-A_3 = 0.$$

The first condition simply implies that the critical Rayleigh number for stationary convection has not yet been attained, and the second condition yields the expression for the Rayleigh numbers at which marginally stable oscillatory modes exist:

$$-R = \frac{(n^2\pi^2+\alpha^2)^2}{\Gamma P\alpha^2}+\frac{1+P}{\Gamma^2P\alpha^2}(n^2\pi^2+\alpha^2). \tag{27}$$

The critical wave-number, obtained by minimizing R with respect to α, is given by

$$\alpha_c^4 = \pi^4+(1+P)\,\pi^2/\Gamma, \tag{28}$$

where consideration has been confined to the lowest-order mode, $n = 1$. The corresponding critical Rayleigh number is

$$-R_c = \frac{\Gamma(\pi^2+\pi\sqrt{(\pi^2+(1+P)/\Gamma)})^2+(1+P)(\pi^2+\pi\sqrt{(\pi^2+(1+P)/\Gamma)})}{\Gamma^2P\pi\sqrt{(\pi^2+(1+P)/\Gamma)}}. \tag{29}$$

After lengthy but straightforward algebra the dimensionless frequency for neutral oscillatory modes is found to be

$$\sigma_c = \pm i\left[\frac{1+P}{\Gamma}(\pi^2+\alpha^2)+\frac{PR\alpha^2}{\pi^2+\alpha^2}\right]^{\frac{1}{2}}. \tag{30}$$

The critical Rayleigh number, wave-number, and frequency for overstability are shown in figures 1, 2 and 3 for several values of Γ and P.

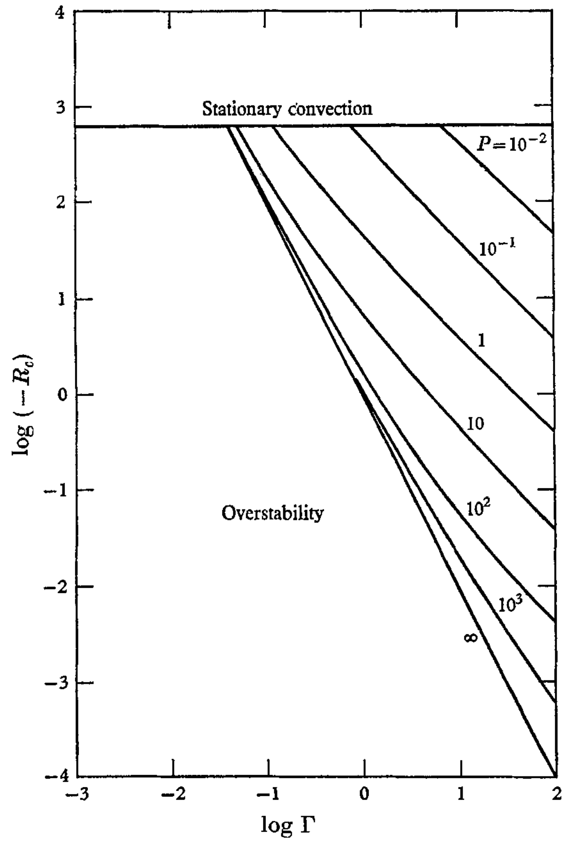

FIGURE 1. Critical Rayleigh numbers for free boundaries.

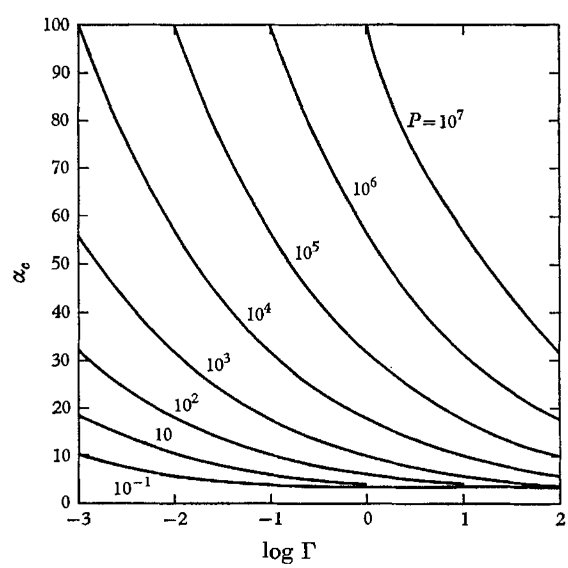

FIGURE 2. Critical wave-numbers for free boundaries.

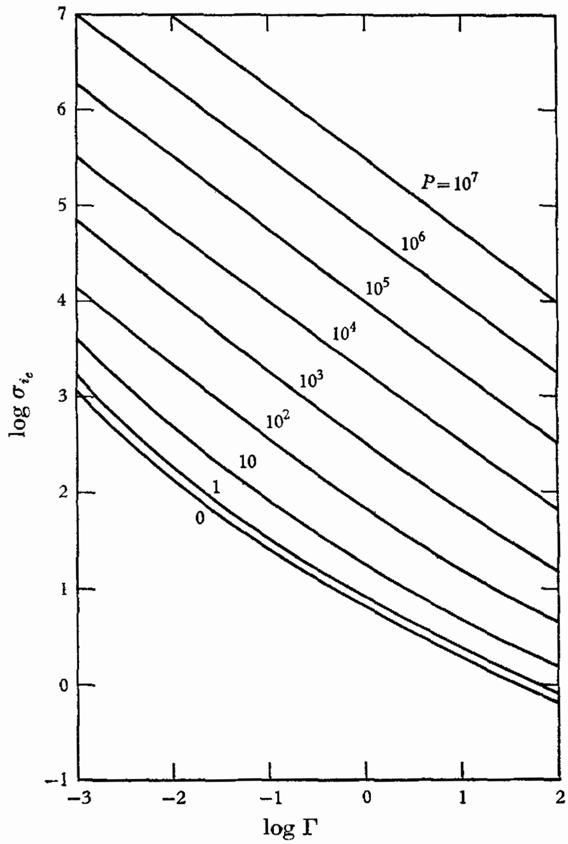

FIGURE 3. Critical frequencies for free boundaries.

It is obvious, on both physical and mathematical grounds, that for neutral stationary modes the solution is identical to that for an ordinary viscous fluid. Hence the critical Rayleigh number and wave-number for stationary convection are $27\pi^4/4$ and $\pi/\sqrt{2}$, respectively (see, for example, Chandrasekhar 1961).

3.2. *Rigid boundaries*

When slip is not allowed at the boundaries the problem is governed by (20) and (21) subject to the conditions

$$W = DW = T = 0 \quad \text{at} \quad z = 0, 1. \tag{31}$$

This differential system considered for marginally stable oscillatory modes, and subject to the physically imposed condition that R be real, constitutes a double eigenvalue problem whose exact solution entails a rather involved numerical procedure. An approximate solution to the problem can, however, be obtained through the use of a variational principle similar to that developed by Chandrasekhar & Elbert (1955) for the solution of the Bénard problem subject to rotation.

Introducing the functions

$$G = (D^2 - \alpha^2)\, W \tag{32}$$

and

$$F = (D^2-\alpha^2)[D^2-\alpha^2-P^{-1}(1+\Gamma\sigma)\sigma]W = [D^2-\alpha^2-P^{-1}(1+\Gamma\sigma)\sigma]G, \quad (33)$$

equations (20) and (21) can be combined and written as

$$(D^2-\alpha^2-\sigma)F = (1+\Gamma\sigma)R\alpha^2 W, \quad (34)$$

and conditions (31) yield

$$W = DW = F = 0 \quad \text{at} \quad z = 0, 1. \quad (35)$$

Multiplication of (34) by F and integration over $[0,1]$ yields, after suitable integrations by parts and utilization of (35),

$$\int_0^1 \{(DF)^2+(\alpha^2+\sigma)F^2\}\,dz = -(1+\Gamma\sigma)R\alpha^2\int_0^1 FW\,dz. \quad (36)$$

After employing (32), (33) and (34) and several integrations by parts, the following expression for R is obtained from (36):

$$-R = \frac{\int_0^1 \{(DF)^2+(\alpha^2+\sigma)F^2\}\,dz}{(1+\Gamma\sigma)\alpha^2\int_0^1 \{G^2+P^{-1}(1+\Gamma\sigma)\sigma[(DW)^2+\alpha^2W^2]\}\,dz}. \quad (37)$$

The condition that the variation δR, R being given by (37), vanish for all small variations δF compatible with the boundary conditions is

$$(D^2-\alpha^2-\sigma)F = (1+\Gamma\sigma)R\alpha^2 W,$$

which is precisely the governing differential equation (34). Hence equation (37) provides the basis for a variational procedure for solving the problem.

For convenience, the origin of the co-ordinate system is translated to the mid-plane and (37) is rewritten in the form

$$-R = \frac{\int_{-\frac{1}{2}}^{+\frac{1}{2}} \{(DF)^2+(\alpha^2+\sigma)F^2\}\,dz}{(1+\Gamma\sigma)\alpha^2\int_{-\frac{1}{2}}^{+\frac{1}{2}} FW\,dz}. \quad (38)$$

Based on experience with related problems (Chandrasekhar 1961) it is anticipated that a value of R accurate to within a few percent can be obtained from (38), with F being approximated by only a single term satisfying (35). Since the most unstable mode for W is expected to be even, this approximate expression is taken to be

$$F = \cos\pi z. \quad (39)$$

Following the procedure developed by Chandrasekhar (1953) an expression for W is derived by solving the equation

$$(D^2-\alpha^2)[D^2-\alpha^2-P^{-1}(1+\Gamma\sigma)\sigma]W = \cos\pi z, \quad (40)$$

obtained by substituting (39) into (33). The solution of this equation which vanishes along with its first derivative at the boundaries $z = \pm\frac{1}{2}$ is

$$W = B_1\cosh\alpha z + B_2\cosh r_1 z + (\cos\pi z)/r_2, \quad (41)$$

where
$$B_1 = -\frac{\pi}{r_2\Delta}\cosh\frac{r_1}{2}, \quad B_2 = \frac{\pi}{r_2\Delta}\cosh\frac{\alpha}{2} \tag{42}$$

in which, for the case of neutral stability,

$$r_1 = [(\alpha^2 - P^{-1}\Gamma\sigma_i^2) + iP^{-1}\sigma_i]^{\frac{1}{2}}, \tag{43}$$

$$r_2 = (\pi^2+\alpha^2)\,[(\pi^2+\alpha^2 - P^{-1}\Gamma\sigma_i^2) + iP^{-1}\sigma_i] \tag{44}$$

and
$$\Delta = r_1\cosh\frac{\alpha}{2}\sinh\frac{r_1}{2} - \alpha\sinh\frac{\alpha}{2}\cosh\frac{r_1}{2}. \tag{45}$$

Evaluation of (38) in terms of the above expressions for F and W yields

$$-R = \frac{\frac{1}{4}r_2(\pi^2+\alpha^2+i\sigma_i)}{\alpha^2(1+i\Gamma\sigma_i)\left[\frac{1}{4}+\dfrac{\pi^2\cosh\frac{1}{2}r_1\cosh\frac{1}{2}\alpha}{\Delta}\left(\dfrac{1}{\pi^2+r_1^2}-\dfrac{1}{\pi^2+\alpha^2}\right)\right]}. \tag{46}$$

In order to determine a state of neutral stability P, Γ, α and σ_i were assigned fixed values and (46) was evaluated numerically. In general this results in a complex value for R. Since R must be real in order to be physically meaningful, a numerical search was conducted to find the value of σ_i for which the imaginary part of R vanishes. When more than one such value was found, the one yielding the lowest real value of $-R$ was, of course, considered. This procedure was then repeated for several values of α in order to trace the neutral curve. For each curve, the procedure was then repeated for several points in the neighbourhood of the critical point, with high accuracy being assured by forcing σ_i to converge to within 0·01 % of the value for which $\text{Im}\,(R) = 0$. After the number of significant figures was estimated by observing the convergence of R, a quadratic curve was fitted to the three points closest to the critical point, and its minimum was taken to represent R_c and α_c. Several critical Rayleigh numbers, wave-numbers and frequencies obtained in this manner for the case of rigid boundaries are given and compared with the corresponding values for free boundaries in table 1.

		R_c		α_c		σ_{ic}	
P	Γ	Free	Rigid	Free	Rigid	Free	Rigid
0·1	1·0	416·49	478·9	3·2256	3·483	6·6088	1·647
1·0	0·1	740·78	877·8	4·1436	4·917	31·799	15·07
1·0	1·0	43·386	51·58	3·2899	3·696	8·004	6·061
10	0·1	198·53	230·0	5·8647	7·309	80·073	76·68
10	1·0	5·9442	7·496	3·7883	4·724	17·368	20·77
100	0·1	123·03	130·1	10·016	11·96	350·00	385·8
100	1·0	1·8689	2·203	5·7514	7·297	66·946	83·45
1000	0·1	106·586	108·0	17·733	20·46	1830·3	2052·0
1000	1·0	1·2205	1·289	9·9942	12·76	333·13	418·8

TABLE 1. Comparison of the critical parameters for two free and two rigid boundaries

Inspection of this table indicates that the effect of rigid boundaries is slightly stabilizing, the maximum increase in R_c due to their presence being only 26 % in this parameter range. R_c for stationary stability, however, increases from 657·5 to 1707·8 (see, for example, Chandrasekhar 1961). Because of the proximity of the curves for the two cases, these results have been tabulated rather than superimposed in figures 1, 2 and 3.

4. Physical significance of the variational principle

Chandrasekhar (1961) has shown that the variational principle for overstable convection under the effect of rotation is equivalent to the thermodynamic relation

$$\partial_t(KE)+\dot{S} = \dot{B}, \tag{47}$$

where KE is the kinetic energy, $\dot{S}$ the rate of dissipation of energy by the viscous stresses, and $\dot{B}$ the rate of liberation of energy by the buoyancy forces in a layer of fluid of height d. The same statement can be made regarding the present problem. To show this, consider first the second term of (47),

$$\dot{S} = -\rho\nu\int_0^d \langle v_i\,\partial_j\tau_{ji}\rangle\,dx_3, \tag{48}$$

where all quantities are dimensional and the angular brackets indicate averaging over the horizontal plane. The shear stresses and velocity are related by (5), from which the expression

$$v_i\,\partial_j(1+t_0\,\partial_t)\,\tau_{ji} = \mu v_i\,\partial_j\,\partial_j\,v_i \tag{49}$$

can be obtained by linearizing and noting that $\partial_i v_i = 0$ under the Boussinesq approximation. Equations (48) and (49) lead to

$$\dot{S} = -\frac{\rho\nu}{1+\Gamma\sigma}\int_0^d \langle v_i\,\nabla^2 v_i\rangle\,dx_3. \tag{50}$$

Introducing dimensioinless velocity profiles representing a simple cellular motion,

$$\left.\begin{aligned} w &= W\exp(\kappa\sigma t/d^2)\cos ax\cos by,\\ u &= -\exp(\kappa\sigma t/d^2)\,(aDW\sin ax\cos by)/\alpha^2,\\ v &= -\exp(\kappa\sigma t/d^2)\,(aDW\cos ax\sin by)/\alpha^2,\end{aligned}\right\} \tag{51}$$

and explicitly carrying out the averaging process, yields

$$\dot{S} = \frac{\rho\nu\kappa^2}{4\alpha^2(1+\Gamma\sigma)\,d}\int_0^1[(D^2-\alpha^2)\,W]^2\,dz. \tag{52}$$

The other two terms, evaluated in the same manner, are found to be

$$\partial_t(KE) = \frac{\rho\kappa^3\sigma}{4\alpha^2 d}\int_0^1\{(DW)^2+\alpha^2W^2\}\,dz \tag{53}$$

and

$$\dot{B} = \frac{\rho\nu\kappa^2}{4\alpha^4(1+\Gamma\sigma)^2\,Rd}\int_0^1 F(D^2-\alpha^2-\sigma)\,F\,dz, \tag{54}$$

which are analogous to those given by Chandrasekhar (1961), apart from differences in non-dimensionalization.

Substitution of (52), (53) and (54) into (47), algebraic rearrangement, and use of the relations $G = (D^2-\alpha^2)\,W$ and

$$\int_0^1 F(D^2F)\,dz = -\int_0^1 (DF)^2\,dz$$

leads to

$$-R = \frac{\int_0^1 \{(DF)^2+(\alpha^2+\sigma)\,F^2\}\,dz}{(1+\Gamma\sigma)\,\alpha^2 \int_0^1 \{G^2+P^{-1}(1+\Gamma\sigma)\,\sigma\,[(DW)^2+\alpha^2W^2]\}\,dz},$$

which is precisely the expression (37) for R, whose minimum was taken to represent the critical state of neutral stability. If instability sets in as stationary convection, the principle reduces to that for the classical Bénard problem, whose significance is discussed by Chandrasekhar (1961). In the overstable case of interest here $\sigma = i\sigma_i$ and the following principle can be stated: overstability will occur at the lowest possible adverse temperature gradient at which the rate of change of kinetic energy can balance, in a synchronous manner, the periodically varying rates of energy dissipation by the shear stresses and energy release by the buoyancy force, assuming that stationary convection has not been initiated.

5. Concluding remarks

The results indicate that the elasticity of a Maxwellian fluid has a destabilizing influence on a liquid layer heated from below. This is true both in the sense that oscillatory convection can occur at a lower critical Rayleigh number than does stationary convection, and that R_c for overstability decreases as Γ increases. The presence of rigid boundaries has a small stabilizing influence on R_c, a small effect on α_c, and a somewhat larger effect on σ_{i_c}.

To discern if this overstability should be observable under laboratory conditions, it is noted from figure 1 that $\Gamma \sim 0{\cdot}1$ is the most favourable condition because most viscoelastic fluids have high Prandtl numbers. In terms of the representative value $\kappa \sim 10^{-3}\,\mathrm{cm^2/sec}$, this condition implies $d^2 \sim 10^{-2}t_0$. An experiment with a layer only 1 mm thick would therefore require a relaxation time of 1 sec. Not only is this value quite large, but in general large relaxation times occur only in very viscous fluids. This in turn necessitates an unreasonably large ΔT to attain the required Rayleigh number. It therefore appears that an experimental investigation under normal laboratory conditions is not feasible. In this regard, however, it should be noted that aqueous solutions of certain recently developed polymers have relatively large relaxation times and rather low viscosities. Perhaps further development of such polymers will make oscillatory convection of more practical concern.

During the final stages of this investigation, a paper by Green (1968) reported another study of overstability in a viscoelastic fluid layer heated from below. His analysis, which is restricted to the case of free boundaries, is carried out in terms of a two time constant model due to Oldroyd. He also concludes that a simple experiment does not appear feasible with currently available fluids.

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